If $\rho \leqslant 0$ and $b > 0$, then $V(u) \leqslant -|\rho| \operatorname{const} |w|^q < 0$. The functional $V(u)$ will have positive values if $\rho \geqslant 0$ and if some of the eigenvalues λ_n of the operator (4. 23) are positive and none are zero. In the latter case we shall have the initial conditions under which the functions $V(u(s, t))$, $|\rho w(t)|$ will be unbounded functions of time.

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PRESSURE OF A PLANE STAMP OF NEARLY CIRCULAR CROSS SECTION ON AN ELASTIC HALF-SPACE

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An approximate method of solving the contact problem of impressing a plane stamp of nearly circular cross section into an elastic half-space is suggested. The friction of the contact surface is neglected. A numerical algorithm for the method is produced. An elliptical and rectangular stamps are considered as examples.

There is no general method of solving the problems for stamps of nearly circular cross section. Apart from the classical problem of a plane elliptical stamp, the literature gives solutions for the problems of polygonal stamps, with each problem however requiring a different approach. An approximate solution for the problem of impressing a stamp of nearly circular cross section into an elastic half-space is given in [1]. The method makes it possible to use the same approach to solve the contact problem for an arbitrary region of contact, and to construct an universal numerical algorithm. The program can be adapted to each particular case by making the corresponding changes in the procedure of computing the Fourier coefficients of the equation of the boundary of the area of contact. Below a numerical algorithm for the approximate method in question is given. A more effective formulation of the solution is given for the case of the elliptical stamp.

1. The equation of the contact region boundary can be written in the form of a Fourier series [1]

$$\rho^2(\varphi) = a^2 + \alpha \sum_{n=1}^{\infty} C_{2n} \cos 2n\varphi = a^2 + \alpha f(\varphi) \tag{1.1}$$

Here α is a small quantity characterizing the deviation of the region of contact from a circle, and a is the radius of this circle.

The pressure under the base of the stamp is obtained in the form of a series in terms of the parameter α (ρ and φ are polar coordinates, and W denotes the settlement of the stamp)

$$P(\rho,\varphi) = \frac{F_0(\rho,\varphi)+\alpha F_1(\rho,\varphi)+\alpha^2 F_2(\rho,\varphi)+\ldots}{\sqrt{\rho^2(\varphi)-\rho^2}} \tag{1.2}$$

$$F_0(\rho,\varphi) = -C, \quad C = W\frac{E}{\pi(1-\nu^2)}$$

$$F_1(\rho,\varphi) = \frac{C}{2}\frac{1}{\rho^2-a^2}\sum_{n=1}^{\infty} C_{2n}\left(1-\frac{\rho^{2n}}{a^{2n}}\right)\cos 2n\varphi = \sum_{n=1}^{\infty} F1_{2n}\cos 2n\varphi$$

$$F_2(\rho,\varphi) = \frac{1}{\rho^2-a^2}\sum_{n=1}^{\infty}(R1_{2n}-R2_{2n})\left(1-\frac{\rho^{2n}}{a^{2n}}\right)\cos 2n\varphi = \sum_{n=1}^{\infty} F2_{2n}\cos 2n\varphi$$

$$R1_{2n}(\rho,\varphi) = \frac{3}{8}C\frac{FF_{2n}}{\rho^2-a^2}\left(1-\frac{\rho^{2n}}{a^{2n}}\right)$$

The quantities FF_{2n} and $R2_{2n}$ can be determined as Fourier coefficients of the functions $f^2\varphi$ and $^1/_2\, F_1(\rho,\varphi)\, f(\varphi)$, respectively.

The numerical algorithm consists of the following steps:

computation of the coefficients C_{2n} of the expansion (1.1) using the formulas [2];

consecutive computation of the one-dimensional blocks $F_{2n} = C_{2n}\cos 2n\varphi$, $F1_{2n}$, FF_{2n}, $R2_{2n}$, $R1_{2n}$ and $F2_{2n}$ using the formulas given above;

computation of the zero, first and second approximation to the pressure using the first formula of (1.2);

blotting the lines of equal pressure.

To find the coefficients FF_{2n} and $R2_{2n}$ we employ the procedure "row (N, $F1$, $F2$, F)" The "row" yields the first N Fourier coefficients of the series $F = F1 \times F2$ according to the expansions

$$F1 = F1_2\cos 2\varphi + F1_4\cos 4\varphi + \ldots + F1_{2N}\cos 2N\varphi$$
$$F2 = F2_2\cos 2\varphi + F2_4\cos 4\varphi + \ldots + F2_{2N}\cos 2N\varphi$$

Every coefficient F_{2L} is determined by the formula

$$F_{2L} = \frac{1}{2}\sum_{i\pm j=L} F1_{2i}\times F2_{2j} \quad (L = 1, 2, \ldots, N)$$

To carry out the last step of the algorithm, it is convenient to begin the computation by covering the region of contact with a grid of concentric circles and polar radii (this is to form the coordinate blocks R and FI) . After this we determine the pressure at the nodes of the polar grid (for $\rho > \rho(\varphi)$, assuming that $P(\rho, \varphi) = 0$) and enter the pressure values in the block $P[I, J']$.

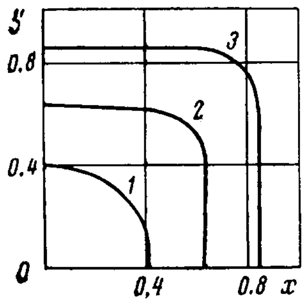

Fig. 1

The lines of equal pressure are constructed in the following order:

a) we fix the value of $P[L, 1]$ ($L = 2, 3, 4, \ldots$) at $FI[1] = 0$;

b) we construct a one-dimensional block $T[J'] = |P[J', K] - P[L, 1]|$ for every $FI[K]$ ($K = 2, 3, 4, \ldots$);

c) we find the smallest element of the block $T[J']$.

The points on the lines of constant pressure are determined as points at which the pressure corresponds to the smallest element of the block T.

The algorithm can be used to solve the problems of arbitrary plane stamps provided that the region of contact is described by the relation (1. 1).

A computing program was written in ALGOL-60 and the computations were carried out on the "M-222" computer.

The pressure was determined in the first two approximations and the lines of constant pressure constructed. The approximate method was checked by applying it to the problem of a plane elliptical stamp. The ellipse semi-axes were $\bar{b} = 1$, $1.1 \leqslant \bar{a} \leqslant 1.4$. Comparison of the numerical results with the exact solution showed that the second approximation values of the pressure differed from the exact values [3] only in the third or fourth decimal place. Five or six terms were sufficient in the series defined by the last two formulas of (1. 2).

Using the above algorithm, we have obtained a numerical solution of the problem of impressing a plane rectangular stamp into an elastic space. Some of the results are given in Table 1 (the sides of the rectangle are: $2\bar{a} = 2.2$, $2\bar{b} = 2$).

In the case of a square stamp, the values of the pressure under the stamp coincide with those obtained in [4]. Figure 1 shows the lines of constant pressure under a square stamp. The curves 1, 2 and 3 correspond to the values of $P / \bar{P}_0$ equal to 1. 08, 1. 21 and 1. 58 ($\bar{P}_0$ is the pressure at the point $\rho = 0$).

Table 1

φ	ρ	P_0/C	P_1/C	P_2/C	φ	ρ	P_0/C	P_1/C	P_2/C
0°	0.0	0.9108	0.8474	0.8487	15°	0.8	1.1989	1.1049	1.1030
0°	0.5	1.0148	0.9236	0.9245	45°	0.0	0.7030	0.8593	0.8566
0°	0.8	1.3153	1.1581	1.1572	45°	0.5	0.7529	0.9538	0.9495
15°	0.0	0.8803	0.8457	0.8464	45°	0.8	0.8463	1.1416	1.1331
15°	0.5	0.9799	0.9258	0.9259					

2. In the case of an elliptical stamp the Fourier coefficients of the equation describing the contact area boundary depend on the parameter α (a and b denote the semi-axes of the ellipse)

$$\rho^2(\varphi) = \left[\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right]^{-1} = a_0^2 + 2a_0^2\alpha\sum_{n=1}^{N}\alpha^{n-1}\cos 2n\varphi$$

$$\alpha = a - b/a + b, \quad a_0^2 = ab$$

The above property of the coefficients makes it possible to alter the final form of the formulas and to obtain the higher order approximations more simply.

Expansion (2.2) of [1] assumes, with the accuracy of up to α^3 the following form:

$$\frac{1}{\sqrt{\zeta - \alpha f(\varphi)}} = \frac{1}{\sqrt{\zeta}}\left\{1 + \alpha\frac{1}{2}\frac{C_2}{\zeta}\cos 2\varphi + \alpha^2\left[\frac{1}{2}\frac{C_4\cos 4\varphi}{\zeta} + \frac{3}{8}\frac{C_2^2\cos^2 2\varphi}{\zeta^2}\right] + \alpha^3\left[\frac{1}{2}\frac{C_6\cos 6\varphi}{\zeta} + \frac{3}{4}\frac{C_2C_4\cos 2\varphi\cos 6\varphi}{\zeta^2} + \frac{5}{16}\frac{C_2^3\cos^3 2\varphi}{\zeta^3} + \ldots\right]\right\}, \quad \zeta = \xi^2 - a_0^2$$

Following the procedure given in [1], we obtain the first three approximations

$$F_0(\rho, \varphi) = -C, \ F_1(\rho, \varphi) = -C\cos 2\varphi, \ F_2(\rho, \varphi) = -{}^3/_4 C\cos 4\varphi$$

$$F_3(\rho, \varphi) = -\frac{1}{8}C\cos 2\varphi + \frac{5}{8a_0^2}C(6\rho^2 - a_0^2)\cos 6\varphi$$

The pressure under the stamp is given by the first formula of (1.2). The particular characteristics of the pressure at the contact area boundary which was assumed from the the start, are retained. It can easily be shown that such representation of the pressure gives good agreement with the exact values [3]. The proposed method can be used to solve the problem of an arbitrary, nearly circular plane stamp.

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